COMPUTING THE GALOIS GROUP OF SOME PARAMETERIZED LINEAR DIFFERENTIAL EQUATION OF ORDER TWO.

THOMAS DREYFUS

ABSTRACT. We extend Kovacic's algorithm to compute the differential Galois group of some second order parameterized linear differential equation. In the case where no Liouvillian solutions could be found, we give a necessary and sufficient condition for the integrability of the system. We give various examples of computation.

CONTENTS

INTRODUCTION

Let us consider the linear differential equation:

$$\begin{pmatrix}\partial_X Y(X)\\ \partial_X^2 Y(X)\end{pmatrix}=\begin{pmatrix}0 & 1\\ r(X) & 0\end{pmatrix}\begin{pmatrix}Y(X)\\ \partial_X Y(X)\end{pmatrix},$$

where $r(X)$ is a rational function with coefficients in $\mathbb{C}$. We have a Galois theory for this type of equation, see [VdPS]. In particular, we can associate to this equation a group H, we call the differential Galois group, that measures the algebraic relations of the solutions. In this case, this group can be viewed as a linear algebraic subgroup of $\mathrm{SL}_2(\mathbb{C})$. Kovacic in [Kov] (see also [VdP]) uses the classification of the linear algebraic subgroup of $\mathrm{SL}_2(\mathbb{C})$ to obtain an algorithm that determines the Liouvillian solutions, which are the solutions that involve exponentials, indefinite integrals and solutions of polynomial equations. In particular, four cases happen:

(1) H is conjugated to a subgroup of $B=\left\{\begin{pmatrix}a & b\\ 0 & a^{-1}\end{pmatrix}, \text{where } a\in\mathbb{C}^*, b\in\mathbb{C}\right\}$ and there exists a Liouvillian solution of the form $e^{\int_0^X f(u)du}$, with $f(X)\in\mathbb{C}(X)$.
(2) H is conjugated to a subgroup of $D_\infty=\left\{\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}\bigcup\begin{pmatrix}0 & b^{-1}\\ -b & 0\end{pmatrix}, \text{where } a,b\in\mathbb{C}^*\right\}$ and there exists a Liouvillian solution of the form $e^{\int_0^X f(u)du}$, where $f(X)$ is algebraic over $\mathbb{C}(X)$ of degree two and $f(X)\notin\mathbb{C}(X)$.
(3) H is finite and all the solutions are algebraic over $\mathbb{C}(X)$.
(4) $H=\mathrm{SL}_2(\mathbb{C})$ and there are no Liouvillian solutions.

Various improvements of this algorithm has been made. See for example [DLR, HVdP, UW, Z]. The case where H is finite has been totally solved in [SU1, SU2], see also [HW].

Let $\{\partial_0,\partial_1,\dots,\partial_n\}$ be a set of $n+1$ commuting derivations. In this article, we are interested in the parameterized linear differential equation of the form:

$$\begin{pmatrix}\partial_0 Y\\ \partial_0^2 Y\end{pmatrix}=\begin{pmatrix}0 & 1\\ r & 0\end{pmatrix}\begin{pmatrix}Y\\ \partial_0 Y\end{pmatrix},$$

where r belongs in a suitable $(\partial_0,\partial_1,\dots,\partial_n)$-differential field. The derivations $\partial_1,\dots,\partial_n$ should be thought as derivations with respect to the parameters. We will denote by C its field of ∂_0-constant. In [L] and [CS, HS], the authors develop a Galois theory for the parameterized linear differential equations. They define a parameterized differential Galois group, that measures the $(\partial_1,\dots,\partial_n)$-differential and algebraic relations between the solutions, see the section 1. This group can be seen as a differential group in the sense of Kolchin: this is a group of matrices whose entries lie in the differential field C and satisfy

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a set of polynomial differential equations in coefficients in C. In the case of the equation $\partial_0^2 Y = rY$, the Galois group will be a linear differential algebraic subgroup of $\mathrm{SL}_2(\mathrm{C})$. The goal of this paper is to extend the algorithm from Kovacic and compute the parameterized differential Galois group of the equation $\partial_0^2 Y = rY$.

The article is presented as follow. In the first section, we recall some basic facts about parameterized differential Galois theory. This theory need to use a field of ∂_0-constant which is $(\partial_1, \dots, \partial_n)$-differentially closed (see [CS] Definition 3.2). We will make a stronger assumption on the field of ∂_0-constant C: we will assume that C is an universal $(\partial_1, \dots, \partial_n)$-field (see the section 1). We do this assumption on C because a field $(\partial_1, \dots, \partial_n)$-differentially closed is an abstract field which has no interpretation as a field of functions. We will see in the section 2 that a result of Seidenberg will allow us to identify the elements of the universal $(\partial_1, \dots, \partial_n)$-field C we will consider as meromorphic functions on a polydisk D of $\mathbb{C}^n$.

In the second section, we recall the result of Seidenberg which implies that the parameterized differential Galois group can be seen as a linear differential algebraic subgroups defined over a field of meromorphic functions on a polydisk D of $\mathbb{C}^n$. Since the original algorithm from [Kov] can be applied if we consider rational function having coefficients in an algebraically closed field, we apply the Kovacic's algorithm for the field of rational function having coefficients in C. We obtain Liouvillian solutions that can be interpreted as meromorphic functions. Then we explain how to compute the Galois group in the four cases of the Kovacic's algorithm. In the case number 4, the Galois group is Zariski dense in SL_2. We recall the definition of an integrable system and the link with integrable system and equations with Galois group that is Zariski dense in SL_2. We decrease the number of integrability conditions by showing that this is enough to check the integrability condition for the pairs of derivations (∂_X, ∂), where ∂ belongs in the vectorial space spanned by the derivations with respect to the parameters. Then, we obtain an effective way to compute the Galois group in the case number 4, see the proposition 9. We summarize the results of the section in the theorem 11.

In the last section we give various examples of computation.

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1. Parameterized differential Galois theory

Let K be a differential field equipped with $n+1$ commuting derivations: $\partial_0, \dots, \partial_n$ and let $\Delta = \{\partial_1, \dots, \partial_n\}$. We will assume that its field of ∂_0-constant C is an universal (Δ)-field with characteristic 0: that is a (Δ)-field such that for any (Δ)-field $C_0 \subset C$, (Δ)-finitely generated over $\mathbb{Q}$, and any (Δ)-finitely generated extension C_1 of C_0, there is a (Δ)-differential C_0-isomorphism of C_1 into C. See [Kol76], Chapter 3, Section 7 for more details. In particular C is (Δ)-differentially closed. In this section, we will recall the result from [CS] of Galois theory for the parameterized linear differential equation of the form:

$$\begin{pmatrix} \partial_0 Y \\ \partial_0^2 Y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ r & 0 \end{pmatrix} \begin{pmatrix} Y \\ \partial_0 Y \end{pmatrix}, \tag{1}$$

with $r \in K$. A parameterized Picard-Vessiot extension of the equation (1) on K is a $(\partial_0, \dots, \partial_n)$-differential field extension $\mathcal{K}|K$ generated over K by the entries of an invertible solution matrix (we will call fundamental solution) and such that the field of ∂_0-constant of $\mathcal{K}$ is equal to C. We can apply the theorem 9.5 of [CS] for the equation (1), and deduce the existence and the unicity up to $(\partial_0, \dots, \partial_n)$-differential isomorphism of the parameterized Picard-Vessiot extension $\mathcal{K}|K$. If $\Delta = \varnothing$, we recover the usual unparameterized Picard-Vessiot extension.

The parameterized (resp. unparameterized) differential Galois group G (resp. H) is the group of field automorphisms of the parameterized Picard-Vessiot extension (resp. the unparameterized Picard-Vessiot extension) of the equation (1), which induces the identity on K and commutes with all the derivations (resp. with the derivation ∂_0). Let U be a fundamental solution. In the unparameterized case,

$$\{U^{-1}\varphi(U), \varphi \in H\},$$

is an linear algebraic subgroup of $\mathrm{GL}_2(\mathrm{C})$. In the parameterized case we find that

$$\{U^{-1}\varphi(U), \varphi \in G\},$$

is a linear differential algebraic subgroup: that is a subgroup of $\mathrm{GL}_2(\mathrm{C})$ which is the zero of a set of (Δ)-differential polynomials in 4 variables. See the theorem 9.10 of [CS] for a proof. Any other fundamental solution yields another differential algebraic subgroup of $\mathrm{GL}_2(\mathrm{C})$ which are all conjugated over $\mathrm{GL}_2(\mathrm{C})$. We will identify G (resp. H) with a linear differential algebraic subgroup of $\mathrm{GL}_2(\mathrm{C})$ (resp. with a linear algebraic subgroup of $\mathrm{GL}_2(\mathrm{C})$) for a chosen fundamental solution. The next lemma is a classical result.

Lemma 1 ([Kov], Section 1.3)**.** $G \subset \mathrm{SL}_2(\mathrm{C})$.

2. Computation of the parameterized differential Galois group

Until the end of the paper, C denotes an universal (Δ)-field equipped with n commuting derivations. Let $C(X)$ be the (∂_X, Δ)-differential field of rational function in the indeterminate X, with coefficients in C, where X is a (Δ)-constant with $\partial_X X = 1$, C is the field of ∂_X-constant and such that ∂_X commutes with all the derivations. Let us consider the parameterized linear differential equation:

$$\begin{pmatrix} \partial_X Y(X) \\ \partial_X^2 Y(X) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ r(X) & 0 \end{pmatrix} \begin{pmatrix} Y(X) \\ \partial_X Y(X) \end{pmatrix}, \tag{2}$$

with $r(X) \in C(X)$. We want to apply Kovacic's algorithm for the parameterized linear differential equation (2). Let $G \subset \mathrm{SL}_2(\mathrm{C})$ be the parameterized differential Galois group. The algorithm from [Kov] can be applied if the field of ∂_X-constant is algebraically closed, which is the case here. The problem is that C is an abstract field which is not very convenient for the computations. In fact we have an interpretation of the elements of C as meromorphic functions. Let C_1 be the (Δ)-differential field generated over $\mathbb{Q}$ by the X-coefficients of $r(X)$. Using the following result of Seidenberg (see [Sei58, Sei69]) with $K_0 = \mathbb{Q}$ and $K_1 = C_1$, we find the existence of a polydisk D of $\mathbb{C}^n$ such that the X-coefficients of $r(X)$ can be considered as meromorphic functions on D.

Theorem 2 (Seidenberg)**.** *Let $\mathbb{Q} \subset K_0 \subset K_1$ be finitely generated (Δ)-differential extensions of $\mathbb{Q}$ and assume that K_0 consists of meromorphic functions on some domain Ω of $\mathbb{C}^n$. Then K_1 is isomorphic to the field K_1^* of meromorphic functions on $\Omega_1 \subset \Omega$, such that $K_0|_{\Omega_1} \subset K_1^*$, and the derivations in Δ can be identified with the derivations with respect to the coordinates on Ω_1.*

Let $(\mathcal{M}_D, \partial_{t_1}, \dots, \partial_{t_n})$ denotes the $\Delta_t = \{\partial_{t_1}, \dots, \partial_{t_n}\}$-differential field of meromorphic functions on D, a polydisk of $\mathbb{C}^n$. Let $t = (t_1, \dots, t_n)$. The discussion above tell us that the $r(X)$ of the equation (2) can be identified with $r(X,t)$, an element of $\mathcal{M}_D(X)$ [1], where D is a polydisk of $\mathbb{C}^n$. We will consider the parameterized linear differential equation:

$$\begin{pmatrix} \partial_X Y(X,t) \\ \partial_X^2 Y(X,t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ r(X,t) & 0 \end{pmatrix} \begin{pmatrix} Y(X,t) \\ \partial_X Y(X,t) \end{pmatrix},$$

with $r(X,t) \in \mathcal{M}_D(X)$. The group G is defined by a finite number of (Δ)-differential polynomial. Again, using the result of Seidenberg with the (Δ)-differential field generated over $\mathbb{Q}$ by the coefficients of the (Δ)-differential polynomial that define G and the X-coefficients of $r(X)$, we deduce that G can be seen as a linear differential algebraic subgroup of $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$. Using again the result of Seidenberg, we remark that after shrinking D, we can assume that if G is conjugated over $\mathrm{SL}_2(\mathrm{C})$ to Q, then we can identify Q and G as linear differential algebraic subgroups of $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$, and they are conjugated over $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$. Furthermore, we obtain that the Liouvillian solutions found are defined over the algebraic closure of $\mathcal{M}_D(X)$. We will compute G as a linear differential algebraic subgroup of $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$. We recall that we will have four cases to consider:

1. There exists a Liouvillian solution of the form: $g(X,t) = e^{\int_0^X f(u,t)du}$, with $f(X,t) \in \mathcal{M}_D(X)$.
2. There exists a Liouvillian solution of the form: $g(X,t) = e^{\int_0^X f(u,t)du}$, where $f(X,t)$ is algebraic over $\mathcal{M}_D(X)$ of degree two and $f(X,t) \notin \mathcal{M}_D(X)$.
3. All the solutions are algebraic over $\mathcal{M}_D(X)$.
4. There are no Liouvillian solutions.

They correspond to the four cases recalled in the introduction. The proposition 6.26 of [HS] says that, if we take the same fundamental solution, the Zariski closure of G is the unparameterized differential Galois group. This means that in each case we are looking at the Zariski dense subgroups of the group given by the usual Kovacic's algorithm.

2.1. We start with the case number 1. There exists a Liouvillian solution of the form:

$$g(X,t) = e^{\int_0^X f(u,t)du},$$

with $f(X,t) \in \mathcal{M}_D(X)$. The action of G on the solution $g(X,t)$ can be computed with the following lemma:

Lemma 3. *Let $\sigma \in G$.*
(1) Let $\alpha(t) \in \mathcal{M}_D$ and $p, q \in \mathbb{N}$, such that $GCD(p,q) = 1$. Then there exists $k \in \mathbb{N}$ such that $\sigma((X-\alpha(t))^{p/q}) = e^{\frac{2ik\pi}{q}}(X-\alpha(t))^{p/q}$.
(2) Let $\alpha(t), \beta(t) \in \mathcal{M}_D$ and $\beta(t) \notin \mathbb{Q}$. Then there exists $a \in \mathbb{C}$ and $c \in \mathbb{C}^*$ such that $\sigma((X-\alpha(t))^{\beta(t)}) = ce^{a\beta(t)}(X-\alpha(t))^{\beta(t)}$.
(3) Let $Q(X,t) \in \mathcal{M}_D(X)$. Then there exists $a \in \mathbb{C}^*$ such that $\sigma(e^{Q(X,t)}) = ae^{Q(X,t)}$.

[1] $\mathcal{M}_D(X)$ denotes the (∂_X, Δ_t)-differential field of rational function with indeterminate X and with coefficients in $\mathcal{M}_D$, such that $\partial_X X = 1$, X is a (Δ_t)-constant and the field $\mathcal{M}_D$ is the field of ∂_X-constant.

Proof. (1) We use the fact the elements of G are fields automorphism that leave invariant $\mathcal{M}_D$.
(2) A computation shows that $\partial_{t_i}(X-\alpha(t))^{\beta(t)} = \left[\log(X-\alpha(t))\partial_{t_i}\beta(t) - \frac{\partial_{t_i}\alpha(t)\beta(t)}{X-\alpha(t)}\right](X-\alpha(t))^{\beta(t)}$. The fact that σ commutes with all the derivations implies the existence of $a \in \mathbb{C}$ and $f(t) \in \mathcal{M}_D$ such that $\sigma(\log(X-\alpha(t)))$ equals to $\log(X-\alpha(t))+a$ and $\sigma((X-\alpha(t))^{\beta(t)}) = f(t)(X-\alpha(t))^{\beta(t)}$. Since $\partial_{t_i}\sigma = \sigma\partial_{t_i}$, we obtain that:

$$\left[\log(X-\alpha(t))\partial_{t_i}\beta(t) + a\partial_{t_i}\beta(t) - \frac{\partial_{t_i}\alpha(t)\beta(t)}{X-\alpha(t)}\right] f(t) = \partial_{t_i}f(t) + f(t)\left[\log(X-\alpha(t))\partial_{t_i}\beta(t) - \frac{\partial_{t_i}\alpha(t)\beta(t)}{X-\alpha(t)}\right].$$

Finally, $f(t)$ satisfies the parameterized linear differential equation

$$\partial_{t_i}\left(\frac{\partial_{t_i}f(t)}{f(t)a\partial_{t_i}\beta(t)}\right) = 0.$$

This means that $\frac{\partial_{t_i}f(t)}{f(t)a\partial_{t_i}\beta(t)} = c \in \mathbb{C}^*$, and $\log f(t) = a\beta(t) + \log(c)$. Then we deduce that $f(t) = ce^{a\beta(t)}$.
(3) We use the fact that

$$\partial_{t_i}\sigma\left(e^{Q(X,t)}\right) = \sigma\left(\partial_{t_i}\left(e^{Q(X,t)}\right)\right) = \sigma\left(\partial_{t_i}\left(Q(X,t)\right)e^{Q(X,t)}\right) = \partial_{t_i}Q(X,t)\sigma\left(e^{Q(X,t)}\right).$$

The equation $\partial_{t_i}\sigma\left(e^{Q(X,t)}\right) = \partial_{t_i}Q(X,t)\sigma\left(e^{Q(X,t)}\right)$ admits $\sigma(e^{Q(X,t)}) = ae^{Q(X,t)}$ with $a \in \mathbb{C}^*$ as solution. □

We deduce that the matrices of G are upper triangular. We will note by $G_m \backsimeq \mathrm{GL}_1(\mathcal{M}_\mathrm{D})$ the multiplicative group. The proof of the following proposition is inspired by the proof of the theorem 1.4 of [Sit]. Let $p : G \to G_m$ that send $\begin{pmatrix} m(t) & a(t) \\ 0 & m^{-1}(t) \end{pmatrix}$, on $m(t)$. Let M be the image of p and $A \subset \mathcal{M}_D$ such that

$$\left\{\begin{pmatrix} 1 & a(t) \\ 0 & 1 \end{pmatrix}, \text{ where } a(t) \in A\right\},$$

is the kernel of p. We have already computed M with the lemma 3. For $m(t) \in M$, let $\Gamma_{m(t)}$ be the set of $\gamma_{m(t)} \in \mathcal{M}_D$ such that $\begin{pmatrix} m(t) & \gamma_{m(t)} \\ 0 & m(t)^{-1} \end{pmatrix} \in G$. We will identify $\mathbb{C}^*$ with the field of constants elements of $\mathcal{M}_D$. If $\mathbb{C}^* \not\subset M$, because of the lemma 3, $g(X,t) \in \mathcal{M}_D(X)$ and we can compute explicitly $g(X,t)\int_{u=0}^{X} g(u,t)^{-2}du$, which is another solution. We obtain explicitly a fundamental solution and we can compute G. The next proposition explain how to compute G when $\mathbb{C}^* \subset M$.

Proposition 4. *Let us keep the same notations. Assume that $\mathbb{C}^* \subset M$. Then G is conjugated to:*

$$\left\{\begin{pmatrix} m(t) & a(t) \\ 0 & m(t)^{-1} \end{pmatrix}, \text{ where } m(t) \in M, a(t) \in A\right\}.$$

For the proof of the proposition, we will need the following lemmas.

Lemma 5. *Assume that $\mathbb{C}^* \subset M$. Let $m(t) \in M$ and $a(t) \in A$. Then $m(t)a(t) \in A$.*

Proof. With the lemma 3, we obtain that for all $m(t) \in M$, there exists $b(t) \in M$ such that $b(t)^2 = m(t)$. Let $m(t) \in M$, $b(t)^2 = m(t)$, $\gamma_{b(t)} \in \Gamma_{b(t)}$, and $a(t) \in A$. The computation:

$$\begin{pmatrix} b(t) & \gamma_{b(t)} \\ 0 & b(t)^{-1} \end{pmatrix}\begin{pmatrix} 1 & a(t) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b(t) & \gamma_{b(t)} \\ 0 & b(t)^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & m(t)a(t) \\ 0 & 1 \end{pmatrix},$$

shows that if $m(t) \in M$ and $a(t) \in A$, then $m(t)a(t) \in A$. □

Lemma 6. *Assume that $\mathbb{C}^* \subset M$. Let $m(t) \in M$. Then $\gamma_{m(t)}, \gamma'_{m(t)} \in \Gamma_{m(t)}$ if and only if $(\gamma_{m(t)} - \gamma'_{m(t)}) \in A$.*

Proof. Let $\gamma_{m(t)}, \gamma'_{m(t)} \in \Gamma_{m(t)}$. The computation:

$$\begin{pmatrix} m(t) & \gamma_{m(t)} \\ 0 & m(t)^{-1} \end{pmatrix}\begin{pmatrix} m(t) & \gamma'_{m(t)} \\ 0 & m(t)^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & m(t)(\gamma_{m(t)} - \gamma'_{m(t)}) \\ 0 & 1 \end{pmatrix},$$

shows that $m(t)(\gamma_{m(t)} - \gamma'_{m(t)}) \in A$, and then $(\gamma_{m(t)} - \gamma'_{m(t)}) \in A$, because of the lemma 5. Conversely, if $(\gamma_{m(t)} - \gamma'_{m(t)}) \in A$ and $\gamma_{m(t)} \in \Gamma_{m(t)}$, then $m(t)(\gamma_{m(t)} - \gamma'_{m(t)}) \in A$, because of the lemma 5. The same computation shows that $\gamma'_{m(t)} \in \Gamma_{m(t)}$. □

Lemma 7. *Assume that $\mathbb{C}^* \subset M$. Let $b \in \mathbb{C}^* \setminus \{\pm 1\}$ and $\gamma_b \in \Gamma_b$. Let:*

$$\beta(t) = b(b^2-1)^{-1}\gamma_b.$$

Then, $\beta(t)(m(t) - m(t)^{-1}) \in \Gamma_{m(t)}$, for all $m(t) \in M$.

Proof. Let $m(t) \in M$ and $\gamma_{m(t)} \in \Gamma_{m(t)}$. The computation:

$$\begin{aligned}&\begin{pmatrix} b & \gamma_b \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} m(t) & \gamma_{m(t)} \\ 0 & m(t)^{-1} \end{pmatrix} \begin{pmatrix} b & \gamma_b \\ 0 & b^{-1} \end{pmatrix}^{-1} \begin{pmatrix} m(t) & \gamma_{m(t)} \\ 0 & m(t)^{-1} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & (1-m(t)^2)b\gamma_b - (1-b^2)m(t)\gamma_{m(t)} \\ 0 & 1 \end{pmatrix},\end{aligned}$$

implies that $(1-m(t)^2)b\gamma_b - (1-b^2)m(t)\gamma_{m(t)} \in A$. Since $(1-b^2)m(t) \in M$, the lemma 5 implies that:

$$(1-b^2)^{-1}m(t)^{-1}(1-m(t)^2)b\gamma_b - \gamma_{m(t)} = \beta(t)(m(t)-m(t)^{-1}) - \gamma_{m(t)} \in A.$$

Therefore $\beta(t)(m(t)-m(t)^{-1}) \in \Gamma_{m(t)}$, because of the lemma 6. □

Proof of the proposition 4. With the lemmas 6 and 7, we find that:

$$G \simeq \left\{ \begin{pmatrix} m(t) & \beta(t)(m(t)-m(t)^{-1}) + a(t) \\ 0 & m(t)^{-1} \end{pmatrix}, \text{ where } m(t) \in M, a(t) \in A \right\}.$$

If we change the fundamental solution (i.e: if we conjugate G over $\mathrm{GL}_2(\mathcal{M}_\mathrm{D})$), we can simplify the expression of G. After a conjugation by the element $P = \begin{pmatrix} 1 & \beta(t) \\ 0 & 1 \end{pmatrix}$, we obtain that:

$$PGP^{-1} \simeq \left\{ \begin{pmatrix} m(t) & a(t) \\ 0 & m(t)^{-1} \end{pmatrix}, \text{ where } m(t) \in M, a(t) \in A \right\}.$$

□

We want now to compute G when $\mathbb{C}^* \subset M$. The computation of M has been already done in the lemma 3. We are now interested to the computation of A, which is a linear differential algebraic subgroup of $(\mathcal{M}_D, +)$. Cassidy classifies the linear differential algebraic subgroups of the additive group in the lemma 11 of [C72]. We define $\mathcal{M}_D[y_1 \dots, y_\nu]_{\Delta_t}$, as the ring of linear homogeneous differential polynomials. There exists $P_1, \dots P_m \in \mathcal{M}_D[y]_{\Delta_t}$, such that:

$$A = \{a(t) \in \mathcal{M}_D | P_1(a(t)) = \cdots = P_m(a(t)) = 0\}.$$

We recall that $g(X,t)\displaystyle\int_{u=0}^{X} g(u,t)^{-2}du$ is another solution. We can choose $\beta(t) \in \mathcal{M}_D$ such that in the basis formed by the solutions $g(X,t)$ and $g(X,t)\displaystyle\int_{u=0}^{X} g(u,t)^{-2}du + \beta(t)g(X,t)$, G is equals to $\left\{ \begin{pmatrix} m(t) & a(t) \\ 0 & m(t)^{-1} \end{pmatrix}, \text{ where } m(t) \in M, a(t) \in A \right\}$. Let $G^g \subset G$ be the subfield of elements that fix $g(X,t)$ and let $\sigma \in G^g$. Let $a(t) \in A$ such that:

$$\sigma\left(g(X,t)\int_{u=0}^{X} g(u,t)^{-2}du + \beta(t)g(X,t)\right) = \left(g(X,t)\int_{u=0}^{X} g(u,t)^{-2}du + \beta(t)g(X,t)\right) + a(t)g(X,t).$$

Since

$$\sigma\left(g(X,t)\int_{u=0}^{X} g(u,t)^{-2}du + \beta(t)g(X,t)\right) = g(X,t)\left(\sigma\left(\int_{u=0}^{X} g(u,t)^{-2}du\right) + \beta(t)\right),$$

we deduce that

$$\sigma\left(\int_{u=0}^{X} g(u,t)^{-2}du\right) - \int_{u=0}^{X} g(u,t)^{-2}du = a(t) \in A.$$

Therefore, the differentials polynomials P_i satisfy $\forall \sigma \in G^g$:

$$\begin{aligned}\sigma\left(P_i\left(\textstyle\int_{u=0}^{X} g(u,t)^{-2}du\right)\right) &= P_i\left(\sigma\left(\textstyle\int_{u=0}^{X} g(u,t)^{-2}du\right)\right) \\ &= P_i\left(\textstyle\int_{u=0}^{X} g(u,t)^{-2}du + a(t)\right) \\ &= P_i\left(\textstyle\int_{u=0}^{X} g(u,t)^{-2}du\right).\end{aligned}$$

Since $P_i\left(\int_{u=0}^{X} g(u,t)^{-2}du\right)$ is fixed by the elements of G^g, we deduce by the Galois correspondence in the parameterized differential Galois theory (see the theorem 9.5 in [CS]), that

$$P_i(a(t)) = 0 \Longleftrightarrow P_i\left(\int_{u=0}^{X} g(u,t)^{-2}du\right) \in \mathcal{M}_D(X)\langle g(X,t)\rangle_{\partial_X,\Delta_t},$$

where $\mathcal{M}_D(X)\langle g(X,t)\rangle_{\partial_X,\Delta_t}$ denotes the $(\partial_X, \Delta_t)-$differential field generated by $\mathcal{M}_D(X)$ and $g(X,t)$.

2.2. Let us consider the case number 2: there exists a Liouvillian solution of the form $e^{\int_0^X f(u,t)du}$, such that $f(X,t)$ satisfies $f(X,t)^2 + a(X,t)f(X,t) + b(X,t) = 0$, where $a(X,t), b(X,t) \in \mathcal{M}_D(X)$. There exists $\varepsilon \in \{\pm 1\}$ such that $f(X,t) = \frac{-a(X,t)+\varepsilon\sqrt{a(X,t)^2-4b(X,t)}}{2}$. By computing the action of G on $e^{\int_0^X \frac{-a(u,t)+\varepsilon\sqrt{a(u,t)^2-4b(u,t)}}{2}du}$, we find that $e^{\int_0^X \frac{-a(u,t)-\varepsilon\sqrt{a(u,t)^2-4b(u,t)}}{2}du}$, is another Liouvillian solution which is linearly independent of the first one. By computing the action of G on the second Liouvillian solution we find the existence of M, a linear differential algebraic subgroup of the multiplicative group G_m such that, in the basis formed by the two Liouvillian solutions:

$$G \simeq \left\{ \begin{pmatrix} a(t) & 0 \\ 0 & a^{-1}(t) \end{pmatrix} \bigcup \begin{pmatrix} 0 & b^{-1}(t) \\ -b(t) & 0 \end{pmatrix}, \text{ where } a(t), b(t) \in M \right\}.$$

We are interested now to the computation of M. A direct computation shows that if there exists $\sigma \in G$ such that $\sigma\left(e^{\int_0^X f(u,t)du}\right) = \alpha(t) e^{\int_0^X f(u,t)du}$, then for all $i \leq n$, $\alpha(t)$ satisfies the parameterized differential equation:

$$\partial_{t_i}\alpha(t) + \alpha(t)\left(\partial_{t_i}\int_0^X f(u,t)du\right) = \alpha(t)\sigma\left(\partial_{t_i}\int_0^X f(u,t)du\right).$$

Let $\widetilde{\partial_{t_i}}\alpha(t) = \frac{\partial_{t_i}\alpha(t)}{\alpha(t)}$ be the logarithm derivation. In [C72] Chapter 4, we see that there exists $P_1, \dots, P_k \in \mathcal{M}_D[y_1 \dots, y_n]_{\Delta_t}$ such that:

$$M \simeq \left\{\alpha(t) | P_1\left(\widetilde{\partial_{t_i}}\alpha(t)\right) = \dots = P_k\left(\widetilde{\partial_{t_i}}\alpha(t)\right) = 0\right\}.$$

The polynomial P_j satisfies, for all $\sigma \in G$, $P_j\left(\partial_{t_i}\int_0^X f(u,t)du\right) = \sigma\left(P_j\left(\partial_{t_i}\int_0^X f(u,t)du\right)\right)$ and then,

$$P_j\left(\widetilde{\partial_{t_i}}\alpha(t)\right) = 0 \Longleftrightarrow P_j\left(\partial_{t_i}\int_0^X f(u,t)du\right) \in \mathcal{M}_D(X).$$

2.3. In the third case, G is finite, because his Zariski closure is finite. Since all finite linear differential algebraic subgroups of $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$ are finite linear algebraic subgroups of $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$, G is equal to the unparameterized differential Galois group. This is the same problem as in the unparameterized case. See [HW] for the computation of G.

2.4. We consider now the case where no Liouvillian solutions are found. We have seen in the introduction that in this case, the unparameterized differential Galois group is $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$. Therefore, G is Zariski dense in $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$.

The classification of the Zariski dense subgroup of $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$ has been made in [C72], Proposition 42. Let $\mathbf{D}$ be the $\mathcal{M}$-vectorial space of derivations of the form:

$$\left\{\sum_{i=0}^{n} a_i(t)\partial_{t_i}, \text{ where } a_i(t) \in \mathcal{M}_D\right\},$$

and $\mathbb{D}$ a vectorial subspace of $\mathbf{D}$. Let $\mathcal{M}_D^{\mathbb{D}}$ be the elements of $\mathcal{M}_D$ that are constant for the derivations in $\mathbb{D}$. Remark that if $\mathbb{D} = \{0\}$, then $\mathcal{M}_D^{\mathbb{D}} = \mathcal{M}_D$. The linear differential algebraic subgroup of $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$ that are Zariski dense in $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$ are conjugated over $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$ to the groups of the form $\mathrm{SL}_2(\mathcal{M}_\mathrm{D}^{\mathbb{D}})$, with $\mathbb{D}$ a vectorial subspace of $\mathbf{D}$.

Let $\mathbb{D} \subset \mathbf{D}$ such that G is conjugated over $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$ to $\mathrm{SL}_2(\mathcal{M}_\mathrm{D}^{\mathbb{D}})$. We want to compute explicitly $\mathbb{D}$. This leads us to the notion of integrable system. Let $A_0(X,t), \dots, A_k(X,t)$, $m \times m$ matrices with entries in $\mathcal{M}_D(X)$ and $\partial'_{t_1}, \dots, \partial'_{t_k} \in \mathbf{D}$. The system

$$[S]: \left\{\begin{array}{rcl} \partial_X Y(X,t) & = & A_0(X,t)Y(X,t) \\ \partial'_{t_1} Y(X,t) & = & A_1(X,t)Y(X,t) \\ & \vdots & \\ \partial'_{t_k} Y(X,t) & = & A_k(X,t)Y(X,t). \end{array}\right.$$

is integrable if and only if, for all $0 \leq i, j \leq k$:

$$\partial'_{t_j} A_i(X,t) - \partial'_{t_i} A_j(X,t) = A_j(X,t)A_i(X,t) - A_i(X,t)A_j(X,t),$$

where $\partial'_{t_0} = \partial_X$. We recall here the proposition 6.3 of [CS], which relates the integrable system and the parameterized differential Galois group in the case where the field of ∂_X-constant is differentially closed.

Proposition 8. *Let $\{\partial'_{t_1},\dots,\partial'_{t_k}\}$ be a commuting basis of $\mathbb{D}$, a vectorial subspace of $\boldsymbol{D}$. G is conjugated to $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}}^{\mathbb{D}})$ over $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}})$ if and only if there exists $A_1(X,t),\dots,A_k(X,t)$, $m\times m$ matrices with entries in $\mathcal{M}_D(X)^2$, such that the system is integrable:*

$$[S]:\left\{\begin{array}{lcl}\partial_X Y(X,t) & = & A(X,t)Y(X,t)\\ \partial'_{t_1} Y(X,t) & = & A_1(X,t)Y(X,t)\\ & \vdots & \\ \partial'_{t_k} Y(X,t) & = & A_k(X,t)Y(X,t).\end{array}\right.$$

We want to give simpler necessary and sufficient condition for the integrability of the system in the proposition 8. First, we will write a necessary and sufficient condition for the integrability of:

$$[S']:\left\{\begin{array}{lcl}\partial_X Y(X,t) & = & A(X,t)Y(X,t)\\ \partial' Y(X,t) & = & A'(X,t)Y(X,t),\end{array}\right.$$

where $A'(X,t)=\begin{pmatrix}a(X,t) & b(X,t)\\ c(X,t) & d(X,t)\end{pmatrix}$ is a $m\times m$ matrix with entries in $\mathcal{M}_D(X)$ and $\partial'\in\mathbf{D}$. The fact that $[S']$ is integrable is equivalent to the solution in $(\mathcal{M}_D(X))^4$ of the parameterized differential system:

$$\left\{\begin{array}{lcl}\partial_X a(X,t) & = & c(X,t)-b(X,t)r(X,t)\\ \partial_X b(X,t) & = & d(X,t)-a(X,t)\\ \partial_X c(X,t) & = & (a(X,t)-d(X,t))r(X,t)+\partial' r(X,t)\\ \partial_X d(X,t) & = & b(X,t)r(X,t)-c(X,t)\end{array}\right.$$

$$\Longleftrightarrow\left\{\begin{array}{lcl}\partial_X a(X,t) & = & -\partial_X d(X,t)\\ \partial_X^2 b(X,t) & = & 2\partial_X d(X,t)\\ \partial_X c(X,t) & = & -\partial_X b(X,t)r(X,t)+\partial' r(X,t)\\ \frac{\partial_X^2 b(X,t)}{2} & = & b(X,t)r(X,t)-c(X,t)\end{array}\right.$$

$$\Longleftrightarrow\left\{\begin{array}{lcl}\partial_X a(X,t) & = & -\partial_X d(X,t)\\ \partial_X^2 b(X,t) & = & 2\partial_X d(X,t)\\ \partial_X c(X,t) & = & -\partial_X b(X,t)r(X,t)+\partial' r(X,t)\\ \frac{\partial_X^3 b(X,t)}{2} & = & 2\partial_X b(X,t)r(X,t)+b(X,t)\partial_X r(X,t)-\partial' r(X,t).\end{array}\right.$$

We can easily see, that the existence of $b(X,t)\in\mathcal{M}_D(X)$ solution of:

$$\frac{\partial_X^3 b(X,t)}{2}=2\partial_X b(X,t)r(X,t)+b(X,t)\partial_X r(X,t)-\partial' r(X,t),$$

is equivalent to the fact that the system $[S']$ is integrable. There exists algorithm to determine if such a system has a solution (see [VdPS] p.100). We obtain a necessary and sufficient condition on ∂' for the integrability condition of the system $[S']$. Let $\mathbb{D}$ be the maximal vectorial subspace of $\mathbf{D}$ such that for all derivations ∂' in $\mathbb{D}$, there exists $A'(X,t)$, $m\times m$ matrix with entries in $\mathcal{M}_D(X)$ such that the following system is integrable:

$$[S']:\left\{\begin{array}{lcl}\partial_X Y(X,t) & = & A(X,t)Y(X,t)\\ \partial' Y(X,t) & = & A'(X,t)Y(X,t).\end{array}\right.$$

We want to prove that the parameterized differential Galois group of $\partial_X Y(X,t)=A(X,t)Y(X,t)$ is conjugated to $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}}^{\mathbb{D}})$ over $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}})$. Assume that this is not the case. Then by the proposition 8, there exists $\mathbb{D}_1,\mathbb{D}_2\subsetneq\mathbb{D}$, having at least dimension 1, with $\mathbb{D}_1\neq\mathbb{D}_2$ such that G is conjugated to $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}}^{\mathbb{D}_1})$ and $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}}^{\mathbb{D}_2})$. In this case, $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}}^{\mathbb{D}_1})$ is conjugated to $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}}^{\mathbb{D}_2})$ over $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}})$. The fact that $\mathbb{D}_1=\mathbb{D}_2$ is proved in [Sit], Theorem 1.2, Chapter 2 but we will recall the proof here. Let $\alpha\in\mathcal{M}_D^{\mathbb{D}_1}$ and consider the diagonal matrix $M=\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\in\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}}^{\mathbb{D}_1})$. Since similar matrices have the same set of eigenvalues and $\mathcal{M}_D^{\mathbb{D}_2}$ is algebraically closed , we obtain that $\alpha(t)\in\mathcal{M}_D^{\mathbb{D}_2}$. Therefore $\mathcal{M}_D^{\mathbb{D}_1}\subset\mathcal{M}_D^{\mathbb{D}_2}$ and by symmetry, $\mathcal{M}_D^{\mathbb{D}_1}=\mathcal{M}_D^{\mathbb{D}_2}$. We deduce then $\mathbb{D}_1=\mathbb{D}_2=\mathbb{D}$. We have proved:

Proposition 9. *We have the following equivalences:*
(1) G is conjugated to $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}}^{\mathbb{D}})$ over $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}})$
(2) For all ∂' that belongs in a commuting basis of $\mathbb{D}$, the following parameterized differential equation has a solution in $\mathcal{M}_D(X)$:

$$\frac{\partial_X^3 b(X,t)}{2}=2\partial_X b(X,t)r(X,t)+b(X,t)\partial_X r(X,t)-\partial' r(X,t).$$

[2] Using the result of Seidenberg, we can identify the matrices as elements of $\mathrm{GL}_2(\mathcal{M}_{\mathrm{D}}(\mathrm{X}))$ because their entries involve a finite number of elements of the fields of ∂_X-constant.

(3) For all $\partial' \in \mathbb{D}$, the following parameterized differential equation has a solution in $\mathcal{M}_D(X)$:

$$\frac{\partial_X^3 b(X,t)}{2} = 2\partial_X b(X,t) r(X,t) + b(X,t)\partial_X r(X,t) - \partial' r(X,t).$$

Remark 10. In the case where $n=1$ (i.e: there is only one parameter) the Zariski dense subgroups of $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}})$ are (up to conjugation over $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}})$) $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}})$ and $\mathrm{SL}_2(\mathbb{C})$. Then we only have to check if:

$$\frac{\partial_X^3 b(X,t)}{2} = 2\partial_X b(X,t) r(X,t) + b(X,t)\partial_X r(X,t) - \partial_t r(X,t),$$

has a solution in $\mathcal{M}_D(X)$ or not.

2.5. We summarize in the next theorem the results of this section.

Theorem 11. *Let us consider $\partial_X^2 Y(X,t) = r(X,t)Y(X,t)$ with $r(X,t) \in \mathcal{M}_D(X)$ and let G be the parameterized differential Galois group, seen as a linear differential algebraic subgroup of $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}})$. There are four possibilities.*

(1) *There exists a Liouvillian solution of the form $g(X,t) = e^{\int_0^X f(u,t)du}$, with $f(X,t) \in \mathcal{M}_D(X)$. There are two possibilities.*

  (a) *If $g(X,t) \in \mathcal{M}_D$, then we can compute explicitly another solution $g(X,t)\displaystyle\int_{u=0}^X g(u,t)^{-2}du$ which linearly independent with $g(X,t)$. In this basis of solution we can compute explicitly G.*

  (b) *In the other case, G, is conjugated to:*

$$\left\{ \begin{pmatrix} m(t) & a(t) \\ 0 & m(t)^{-1} \end{pmatrix}, \text{ where } m(t) \in M, a(t) \in A \right\},$$

where:

$M = \{f(t) \in \mathcal{M}_D | \exists \sigma \in G \text{ such that } g(X,t)^{-1}\sigma(g(X,t))\}$.

$A = \left\{ a(t) \in \mathcal{M}_D | \forall P \in \mathcal{M}_D[y]_{\Delta_t}, P\left(\int_{u=0}^X g(u,t)^{-2}du\right) \in \mathcal{M}_D(X)\langle g(X,t)\rangle_{\partial_X,\Delta_t} \iff P(a(t)) = 0 \right\}$.

(2) *There exists a Liouvillian solution of the form $g(X,t) = e^{\int_0^X f(u,t)du}$, where $f(X,t)$ is algebraic over $\mathcal{M}_D(X)$ of degree two and $f(X,t) \notin \mathcal{M}_D(X)$. In this case, G is conjugated to*

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \bigcup \begin{pmatrix} 0 & b^{-1} \\ -b & 0 \end{pmatrix}, \text{ where } a,b \in M \right\}, \text{ where,}$$

$$M = \left\{ f(t) \in \mathcal{M}_D | \forall P \in \mathcal{M}_D[y_1 \dots, y_n]_{\Delta_t}, P\left(\partial_{t_i}\int_{u=0}^X f(u,t)du\right) \in \mathcal{M}_D(X) \Leftrightarrow P\left(\widetilde{\partial}_{t_i} f(t)\right) = 0 \right\}.$$

(3) *G is finite. In this case, G is equals to the unparameterized differential Galois group.*

(4) *There are no Liouvillian solutions. In this case, there exists $\mathbb{D}$, a $\mathcal{M}_D$-vectorial space of derivations spanned by Δ_t, such that G is conjugated to $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}}^{\mathbb{D}})$. Moreover, $\partial_t' \in \mathbb{D}$ if and only if the following parameterized differential equation has a solution in $\mathcal{M}_D(X)$:*

$$\frac{\partial_X^3 b(X,t)}{2} = 2\partial_X b(X,t) r(X,t) + b(X,t)\partial_X r(X,t) - \partial_t' r(X).$$

Notice that the computation of the Liouvillian solutions and the unparameterized differential Galois group are already known. Our results compute the parameterized differential Galois group in the cases 1,2 and 4. The classification of the Zariski dense linear differential algebraic subgroup of $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}})$ and the link with integrable system where already known (see [C72, CS]), but we give here an effective way to compute the Galois group in the case number 4 and we decrease the number of integrability conditions.

3. Examples

In the following examples, we will consider equations having coefficients in $\mathcal{M}_D(X)$ and we will compute G as a linear differential algebraic subgroup of $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}})$. In the three first examples, we are in the case where no Liouvillian solutions are found. In the fourth example, we are in the case number 1 and in the last example, we are in the case number 2.

Example 12 (Schrodinger equation with rational potential of odd degree). Let $r(X,t) = X^{2n+1} + \displaystyle\sum_{i=0}^{2n} t_i X^i$.

There are no Liouvillian solutions. The parameterized linear differential equation:

$$\frac{\partial_X^3 b(X,t)}{2} = 2\partial_X b(X,t) r(X,t) + b(X,t)\partial_X r(X,t) - \sum_{i=0}^{2n} a_i(t) X^i,$$

has a rational solution if and only if there exists $c(t) \in \mathcal{M}_D$ such that:

$$\begin{cases} & a_{2n+1}(t) & \in & \mathcal{M}_D \\ & a_{2n}(t) & = & c(t)(2n+1) \\ i < 2n: & a_i(t) & = & c(t)(i+1)t_{i+1}. \end{cases}$$

And then:

$$G \simeq \mathrm{SL}_2(\mathcal{M}_{\mathrm{D}}^{\partial_{t'_1}, \partial_{t'_2}}),$$

where:

$$\partial_{t'_1} = (2n+1)\partial_{t_{2n}} + \sum_{i=0}^{2n-1} (i+1)t_{i+1}\partial_{t_i} \text{ and } \partial_{t'_2} = \partial_{t_{2n+1}}.$$

Example 13 (Bessel equation). Let $r(X,t) = \frac{4t^2-1}{4X^2} - 1$. In [Kov], Example 2 of the section 4.2, we see that if $t \notin \frac{1}{2} + \mathbb{Z}$, this parameterized linear differential equation has no Liouvillian solution. We can choose D such that $\{D \cap (\frac{1}{2} + \mathbb{Z})\} = \varnothing$. We obtain that G is Zariski dense in $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}})$. With the remark 10, we have to see if the parameterized linear differential equation:

$$\frac{\partial_X^3 b(X,t)}{2} = 2\partial_X b(X,t)\left(\frac{4t^2-1}{4X^2} - 1\right) + b(X,t)\frac{1-4t^2}{2X^3} - \frac{2t}{X}, \tag{3}$$

has a solution in $\mathcal{M}_D(X)$ or not. Suppose that there exists $b(X,t) \in \mathcal{M}_D(X)$ satisfying such an equation. We can see directly that if $b(X,t)$ has a poles, then it is $X = 0$. Assume that $b(X,t)$ has a poles of order ν at $X = 0$ and let $0 \neq f(t) \in \mathcal{M}_D$ equals to the value at $(0,t)$ of $X^\nu b(X,t)$. Since $b(X,t)$ satisfy the equation (3), we find for all $t \in D$:

$$\frac{-f(t)\nu(\nu-1)(\nu-2)}{2} = -f(t)\nu\frac{4t^2-1}{2} + f(t)\frac{1-4t^2}{2}.$$

For all ν, there is no $0 \neq f(t)$ satisfying this equality and we find that $b(X,t) \in \mathcal{M}_D[X]$. Let ν its degree and $f(t)$ its leading term. The equation (3) has no constant solution and we can assume $\nu > 1$. We find for all $t \in D$:

$$0 = -2\nu f(t),$$

which implies that the equation (3) has no solutions in $\mathcal{M}_D(X)$ and then:

$$G \simeq \mathrm{SL}_2(\mathcal{M}_{\mathrm{D}}).$$

Example 14 (Harmonic oscillator). Let $r(X,t) = \frac{X^2}{4} + t$. There are no Liouvillian solution. With the remark 10, we have to check if the parameterized linear differential equation:

$$\frac{\partial_X^3 b(X,t)}{2} = 2\partial_X b(X,t)\left(\frac{X^2}{4} + t\right) + b(X,t)\frac{X}{2} - 1,$$

has a solution in $\mathcal{M}_D(X)$ or not. We can see directly that if $b(X,t) \in \mathcal{M}_D(X)$ is solution, then it has no poles, which means that $b(X,t) \in \mathcal{M}_D[X]$. Let ν be its degree and $0 \neq f(t)$ its leading term. We find that $\frac{(\nu+1)f(t)}{2} = 0$, which admit no solution different from 0. Then:

$$G \simeq \mathrm{SL}_2(\mathcal{M}_{\mathrm{D}}).$$

Example 15. If $r(X,t) = \frac{t}{X^2}$, then we have two Liouvillian solutions:

$$f_1(X,t) = \sqrt{X}X^{\frac{\sqrt{1+4t}}{2}} \text{ and } f_2(X,t) = \sqrt{X}X^{-\frac{\sqrt{1+4t}}{2}}.$$

We can compute the parameterized differential Galois group, for the fundamental solution

$$\begin{pmatrix} f_1(X,t) & f_2(X,t) \\ \partial_X f_1(X,t) & \partial_X f_2(X,t) \end{pmatrix}:$$

$$G \simeq \left\{ \begin{pmatrix} \alpha e^{a(\sqrt{1+4t})} & 0 \\ 0 & \alpha^{-1} e^{-a(\sqrt{1+4t})} \end{pmatrix}, \text{ where } a \in \mathbb{C}, \alpha \in \mathbb{C}^* \right\}.$$

Viewed as a linear differential algebraic subgroup $\mathrm{GL}_2(\mathcal{M}_{\mathrm{D}})$,

$$G \simeq \left\{ \begin{pmatrix} \alpha(t) & 0 \\ 0 & \alpha^{-1}(t) \end{pmatrix}, \text{ where } \partial_t \left(\frac{\sqrt{1+4t}\partial_t \alpha(t)}{\alpha(t)} \right) = 0 \right\}.$$

Example 16. If $r(X,t) = \frac{t}{X} - \frac{3}{16X^2}$, then we have two Liouvillian solutions:

$$f_1(X,t) = (X)^{1/4} e^{2(tX)^{1/2}} \text{ and } f_2(X,t) = (X)^{1/4} e^{-2(tX)^{1/2}}.$$

We can compute the parameterized differential Galois group, for the fundamental solution

$$\begin{pmatrix} f_1(X,t) & f_2(X,t) \\ \partial_X f_1(X,t) & \partial_X f_2(X,t) \end{pmatrix}:$$

$$G \simeq \left\{ \begin{pmatrix} a(t) & 0 \\ 0 & a^{-1}(t) \end{pmatrix} \bigcup \begin{pmatrix} 0 & b^{-1}(t) \\ -b(t) & 0 \end{pmatrix}, \text{ where } a(t), b(t) \in \mathbb{C}^* \right\}.$$

We can remark that we have an integrable system:

$$\left\{ \begin{array}{rcl} \partial_X Y(X,t) & = & A(X,t)Y(X,t) \\ \partial_t Y(X,t) & = & B(X,t)Y(X,t) \end{array} \right.$$

with:

$$A(X,t) = \begin{pmatrix} 0 & 1 \\ \frac{t}{X} - \frac{3}{16X^2} & 0 \end{pmatrix} \text{ and } B(X,t) = \begin{pmatrix} 0 & \frac{X}{t} \\ 1 - \frac{3}{16tX} & \frac{3}{4t} \end{pmatrix}.$$

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Institut de mathématiques de Jussieu, Topologie et géométrie algébriques,
Current address: 4, place Jussieu 75005 Paris.
E-mail address: `tdreyfus@math.jussieu.fr`.